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Birational Arakelov geometry

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Birational Arakelov Geometry

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Birational Arakelov Geometry

Problem:

For a real number $\lambda > 1$, find an asymptotic estimate of

$$\log \# \{ (a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n} \}$$

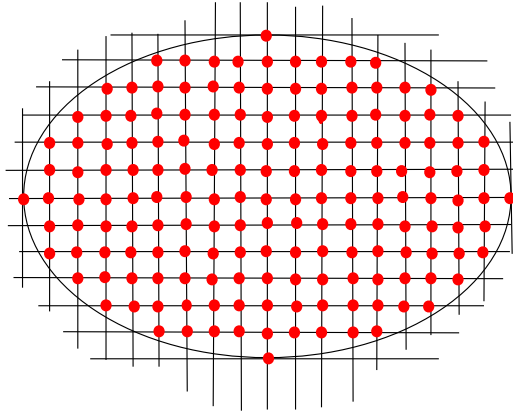
with respect to n .



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How many lattice points in the ellipse?



$$a^2 + 2b^2 \leq \lambda^{2n}$$

Considering a shrinking map $(x, y) \mapsto (\lambda^{-n}x, \lambda^{-n}y)$,

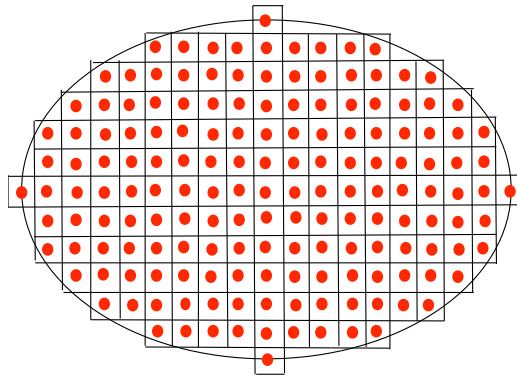
$$\begin{aligned} \# \{ (a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n} \} \\ = \# \{ (a', b') \in (\mathbb{Z}\lambda^{-n})^2 \mid a'^2 + 2b'^2 \leq 1 \}. \end{aligned}$$

We assign a square

$$\left[a' - \frac{\lambda^{-n}}{2}, a' + \frac{\lambda^{-n}}{2} \right] \times \left[b' - \frac{\lambda^{-n}}{2}, b' + \frac{\lambda^{-n}}{2} \right]$$

to each element of

$$\{ (a', b') \in (\mathbb{Z}\lambda^{-n})^2 \mid a'^2 + 2b'^2 \leq 1 \}.$$



$$x^2 + 2y^2 \leq 1$$

\sum (the volume of each square) \sim the volume of the ellipse

Thus

$$\begin{aligned} & \# \{ (a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n} \} \times (\lambda^{-n})^2 \\ & \sim \text{the volume of } \{ (x, y) \in \mathbb{R}^2 \mid x^2 + 2y^2 \leq 1 \} = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

Therefore,

$$\log \# \{ (a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n} \} \sim (2 \log \lambda) n.$$

Let K be a number field (i.e. a finite extension of \mathbb{Q}) and let $K(\mathbb{C})$ be the set of all embeddings $K \hookrightarrow \mathbb{C}$. Note that $\#(K(\mathbb{C})) = [K : \mathbb{Q}]$ and $K(\mathbb{C})$ is the set of \mathbb{C} -valued points of $\text{Spec}(K)$. Let O_K be the ring of integers in K , that is,

$$O_K = \{x \in K \mid x \text{ is integral over } \mathbb{Z}\}.$$

We set $X = \text{Spec}(O_K)$. Let $\text{Div}(X)$ be the group of divisors on X , that is,

$$\text{Div}(X) := \bigoplus_{P \in X \setminus \{0\}} \mathbb{Z}[P].$$

For $D = \sum_P a_P [P]$, $\deg(D)$ is defined by

$$\deg(D) := \sum_P a_P \log \#(O_K/P).$$

$\widehat{\text{Div}}(X)$ is defined by

$$\widehat{\text{Div}}(X) = \text{Div}(X) \times \{\xi \in \mathbb{R}^{K(\mathbb{C})} \mid \xi_\sigma = \xi_{\bar{\sigma}} \ (\forall \sigma \in K(\mathbb{C}))\},$$

where $\bar{\sigma}$ is the composition of $\sigma : K \hookrightarrow \mathbb{C}$ and the complex conjugation $\mathbb{C} \xrightarrow{\bar{}} \mathbb{C}$. An element of $\widehat{\text{Div}}(X)$ is called an [arithmetic divisor](#) on X . For simplicity, an element $\xi \in \mathbb{R}^{K(\mathbb{C})}$ is denoted by $\sum_\sigma \xi_\sigma [\sigma]$. For example, if we set

$$(\widehat{x}) := \left(\sum_P \text{ord}_P(x) [P], \sum_\sigma -\log |\sigma(x)|^2 [\sigma] \right)$$

for $x \in K^\times$, then $(\widehat{x}) \in \widehat{\text{Div}}(X)$, which is called an [arithmetic principal divisor](#).

The **arithmetic degree** $\widehat{\deg}(\overline{D})$ for $\overline{D} = (D, \xi)$ is defined by

$$\widehat{\deg}(\overline{D}) := \deg(D) + \frac{1}{2} \sum_{\sigma} \xi_{\sigma}.$$

Note that $\widehat{\deg}(\widehat{(x)}) = 0$ by the product formula. For

$$\overline{D} = \left(\sum_P n_P [P], \sum_{\sigma} \xi_{\sigma} [\sigma] \right),$$

$$\overline{D} \geq 0 \iff^{\text{def}} n_P \geq 0 \text{ and } \xi_{\sigma} \geq 0 \text{ for all } P \text{ and } \sigma$$

We set

$$\hat{H}^0(X, \overline{D}) := \{x \in K^{\times} \mid \overline{D} + \widehat{(x)} \geq 0\} \cup \{0\}.$$

Set $K = \mathbb{Q}(\sqrt{-2})$. Then $O_K = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$ and $K(\mathbb{C}) = \{\sigma_1, \sigma_2\}$ given by $\sigma_1(\sqrt{-2}) = \sqrt{-2}$ and $\sigma_2(\sqrt{-2}) = -\sqrt{-2}$. We set $\overline{D} = (0, \log(\lambda^2)[\sigma_1] + \log(\lambda^2)[\sigma_2])$. Then $\widehat{\deg}(\overline{D}) = 2 \log(\lambda)$. Note that, for $x = a + b\sqrt{-2} \in \mathbb{Q}(\sqrt{-2}) \setminus \{0\}$,

$$\begin{aligned} n\overline{D} + \widehat{(x)} \geq 0 &\iff \begin{cases} n \log(\lambda^2) - \log(a^2 + 2b^2) \geq 0 \\ a, b \in \mathbb{Z} \end{cases} \\ &\iff \begin{cases} a^2 + 2b^2 \leq \lambda^{2n} \\ a, b \in \mathbb{Z} \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned}\hat{H}^0(X, n\bar{D}) &= \left\{ x \in K^\times \mid n\bar{D} + \widehat{(x)} \geq 0 \right\} \cup \{0\} \\ &= \{ a + b\sqrt{-2} \in \mathbb{Z}[\sqrt{-2}] \mid a^2 + 2b^2 \leq \lambda^{2n} \}.\end{aligned}$$

Thus the previous observation means that

$$\log \# \hat{H}^0(X, n\bar{D}) \sim \widehat{\deg}(\bar{D})n.$$

Theorem (Arithmetic Hilbert-Samuel formula for $\text{Spec}(O_K)$)

If $\widehat{\deg}(\bar{D}) > 0$, then $\log \# \hat{H}^0(n\bar{D}) = n\widehat{\deg}(\bar{D}) + O(1)$. In particular, if $n \gg 1$, then there is $x \in K^\times$ with $n\bar{D} + \widehat{(x)} \geq 0$. Moreover, $\lim_{n \rightarrow \infty} \log \# \hat{H}^0(n\bar{D})/n = \widehat{\deg}(\bar{D})$.

Remark

Let r_2 be the number of complex embeddings K into \mathbb{C} and let D_K be the discriminant of K over \mathbb{Q} . If

$$\widehat{\deg}(\bar{D}) \geq \log((\pi/2)^{r_2} \sqrt{|D_K|}),$$

then $\hat{H}^0(\bar{D}) \neq \{0\}$.

Indeed, we set $\bar{D} = (0, \xi)$. As $\widehat{\deg}(\bar{D}) = 0$, there are $x \in K_{\mathbb{R}}^{\times}$ such that $\bar{D} + (\widehat{x})_{\mathbb{R}} \geq 0$. Note that $\widehat{\deg}(\bar{D} + (\widehat{x})_{\mathbb{R}}) = 0$, so that

$$\bar{D} + (\widehat{x})_{\mathbb{R}} = (0, 0).$$

On the other hand, we can find $x_1, \dots, x_r \in K^{\times}$ and $a_1, \dots, a_r \in \mathbb{R}$ such that $x = x_1^{a_1/2} \cdots x_r^{a_r/2}$ and a_1, \dots, a_r are linearly independent over \mathbb{Q} . Thus,

$$\begin{cases} \sum_{i=1}^r a_i \operatorname{ord}_P(x_i) = 0 & \text{for all } P \\ \xi_{\sigma} = \sum_{i=1}^r a_i \log |\sigma(x_i)| & \text{for all } \sigma \end{cases}$$

Using the linear independency of a_1, \dots, a_r over \mathbb{Q} , we have $\operatorname{ord}_P(x_i) = 0$ for all P and i . This means that $x_i \in O_K^{\times}$ for all i , as required.

Remark

The above theorem does not hold on an algebraic curve. In this sense, it is a purely arithmetic problem.

Let M be an n -equidimensional smooth projective variety over \mathbb{C} .

Let $\operatorname{Div}(M)$ be the group of Cartier divisors on M and let

$\operatorname{Div}(M)_{\mathbb{R}} := \operatorname{Div}(M) \otimes_{\mathbb{Z}} \mathbb{R}$, whose element is called an **\mathbb{R} -divisor**.

Let us fix $D \in \operatorname{Div}(M)_{\mathbb{R}}$. We set $D = a_1 D_1 + \cdots + a_l D_l$, where $a_1, \dots, a_l \in \mathbb{R}$ and D_i 's are prime divisors on M .

Let $g : M \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be a locally integrable function on M .

We say g is a **D -Green function of C^{∞} -type (resp. C^0 -type)** if, for each point $x \in M$, there are an open neighborhood U_x of x , local equations f_1, \dots, f_l of D_1, \dots, D_l respectively and a C^{∞} (resp. C^0) function u_x over U_x such that

$$g = u_x + \sum_{i=1}^l (-a_i) \log |f_i|^2 \quad (\text{a.e.})$$

over U_x . The above equation is called a **local expression** of g .

Let g be a D -Green function of C^0 -type on M . Let

$$g = u + \sum (-a_i) \log |f_i|^2 = u' + \sum (-a_i) \log |f'_i|^2 \quad (a.e.)$$

be two local expressions of g . Then, as $\sum (-a_i) \log |f_i/f'_i|^2$ is dd^c -closed, we have $dd^c(u) = dd^c(u')$ as currents, so that it can be defined globally. We denote it by $c_1(D, g)$. Note that $c_1(D, g)$ is a closed $(1, 1)$ -current on M . If g is of C^∞ -type, then $c_1(D, g)$ is represented by a C^∞ -form.

Let X be a d -dimensional, generically smooth normal projective arithmetic variety, that is,

- ① X is projective flat integral scheme over \mathbb{Z} .
- ② If $X_{\mathbb{Q}} = X \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q})$ is the generic fiber of $X \rightarrow \text{Spec}(\mathbb{Z})$, then $X_{\mathbb{Q}}$ is smooth over \mathbb{Q} .
- ③ The Krull dimension of X is d , that is, $\dim X_{\mathbb{Q}} = d - 1$.
- ④ X is normal.

Let $\text{Div}(X)$ be the group of Cartier divisors on X and $\text{Div}(X)_{\mathbb{R}} = \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, whose element is called an **\mathbb{R} -divisor** on X . For $D \in \text{Div}(X)_{\mathbb{R}}$, we set $D = \sum_i a_i D_i$, where $a_i \in \mathbb{R}$ and D_i 's are reduced and irreducible subschemes of codimension one. We say D is **effective** if $a_i \geq 0$ for all i . Moreover, for $D, E \in \text{Div}(X)_{\mathbb{R}}$,

$$D \leq E \text{ (or } E \geq D) \iff E - D \text{ is effective}$$

Let D be an \mathbb{R} -divisor on X and let g be a locally integrable function on $X(\mathbb{C})$. We say a pair $\bar{D} = (D, g)$ is an **arithmetic \mathbb{R} -divisor** on X if $F_\infty^*(g) = g$ (a.e.), where $F_\infty : X(\mathbb{C}) \rightarrow X(\mathbb{C})$ is the complex conjugation map, i.e. for $x \in X(\mathbb{C})$, $F_\infty(x)$ is given by the composition $\text{Spec}(\mathbb{C}) \xrightarrow{\bar{\cdot}} \text{Spec}(\mathbb{C}) \xrightarrow{\chi} X$. Moreover, we say \bar{D} is of **C^∞ -type (resp. C^0 -type)** if g is a D -Green function of C^∞ -type (resp. C^0 -type). For arithmetic divisors $\bar{D}_1 = (D_1, g_1)$ and $\bar{D}_2 = (D_2, g_2)$, we define $\bar{D}_1 = \bar{D}_2$ and $\bar{D}_1 \leq \bar{D}_2$ to be

$$\begin{aligned}\bar{D}_1 = \bar{D}_2 &\iff D_1 = D_2 \text{ and } g_1 = g_2 \text{ (a.e.)}, \\ \bar{D}_1 \leq \bar{D}_2 &\iff D_1 \leq D_2 \text{ and } g_1 \leq g_2 \text{ (a.e.)}.\end{aligned}$$

We say \bar{D} is effective if $\bar{D} \geq (0, 0)$.

Let $\text{Rat}(X)$ be the field of rational functions on X . For $\phi \in \text{Rat}(X)^\times$, we set

$$(\phi) := \sum_{\Gamma} \text{ord}_{\Gamma}(\phi) \Gamma \quad \text{and} \quad \widehat{(\phi)} := ((\phi), -\log |\phi|^2).$$

Note that $\widehat{(\phi)}$ is an arithmetic divisor of C^∞ -type

Let $\overline{D} = (D, g)$ be an arithmetic \mathbb{R} -divisor of C^0 -type on X .

- $H^0(X, D) := \{\phi \in \text{Rat}(X)^\times \mid D + (\phi) \geq 0\} \cup \{0\}$. Note that $H^0(X, D)$ is finitely generated \mathbb{Z} -module.
- $\hat{H}^0(X, \overline{D}) := \{\phi \in \text{Rat}(X)^\times \mid \overline{D} + (\widehat{\phi}) \geq (0, 0)\} \cup \{0\}$. Note that $\hat{H}^0(X, \overline{D})$ is a finite set.
- $\hat{h}^0(X, \overline{D}) := \log \# \hat{H}^0(X, \overline{D})$.
- $\widehat{\text{vol}}(\overline{D}) := \limsup_{n \rightarrow \infty} \frac{\log \# \hat{H}^0(X, n\overline{D})}{n^d/d!}$.

Theorem

- 1 $\widehat{\text{vol}}(\overline{D}) < \infty$.
- 2 (H. Chen) $\widehat{\text{vol}}(\overline{D}) := \lim_{n \rightarrow \infty} \frac{\log \# \hat{H}^0(X, n\overline{D})}{n^d/d!}$.
- 3 $\widehat{\text{vol}}(a\overline{D}) = a^d \widehat{\text{vol}}(\overline{D})$ for $a \in \mathbb{R}_{\geq 0}$.
- 4 (Moriwaki) $\widehat{\text{vol}} : \widehat{\text{Div}}_{C^0}(X)_{\mathbb{R}} \rightarrow \mathbb{R}$ is continuous in the following sense: Let $\overline{D}_1, \dots, \overline{D}_r, \overline{A}_1, \dots, \overline{A}_s$ be arithmetic \mathbb{R} -divisors of C^0 -type on X . For a compact subset B in \mathbb{R}^r and a positive number ϵ , there are positive numbers δ and δ' such that

$$\left| \widehat{\text{vol}} \left(\sum a_i \overline{D}_i + \sum \delta_j \overline{A}_j + (0, \phi) \right) - \widehat{\text{vol}} \left(\sum a_i \overline{D}_i \right) \right| \leq \epsilon$$

for all $a_1, \dots, a_r, \delta_1, \dots, \delta_s \in \mathbb{R}$ and $\phi \in C^0(X)$ with $(a_1, \dots, a_r) \in B$, $|\delta_1| + \dots + |\delta_s| \leq \delta$ and $\|\phi\|_{\text{sup}} \leq \delta'$.

Let C be a reduced and irreducible 1-dimensional closed subscheme of X . We would like to define $\widehat{\deg}(\overline{D}|_C)$. It is characterized by the following properties:

- ① $\widehat{\deg}(\overline{D}|_C)$ is linear with respect to \overline{D} .
- ② If $\phi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$, then $\widehat{\deg}(\widehat{(\phi)}_{\mathbb{R}}|_C) = 0$.
- ③ If $C \not\subseteq \text{Supp}(D)$ and C is vertical, then $\widehat{\deg}(\overline{D}|_C) = \log(p) \deg(D|_C)$, where C is contained in the fiber over a prime p .
- ④ If $C \not\subseteq \text{Supp}(D)$ and C is horizontal, then $\widehat{\deg}(\overline{D}|_C) = \widehat{\deg}(\overline{D}|_{\tilde{C}})$, where \tilde{C} is the normalization of C . Note that $\tilde{C} = \text{Spec}(O_K)$ for some number field K .

- \overline{D} is **big** $\iff \widehat{\text{vol}}(\overline{D}) > 0$.
- \overline{D} is **psedo-effective** $\iff \overline{D} + \overline{A}$ is big for any big arithmetic \mathbb{R} -divisor \overline{A} of C^0 -type.
- $\overline{D} = (D, g)$ is **nef** \iff
 - ① $\widehat{\deg}(\overline{D}|_C) \geq 0$ for all reduced and irreducible 1-dimensional closed subschemes C of X .
 - ② $c_1(D, g)$ is a positive current.
- $\overline{D} = (D, g)$ is **relatively nef** \iff
 - ① $\widehat{\deg}(\overline{D}|_C) \geq 0$ for all **vertical** reduced and irreducible 1-dimensional closed subschemes C of X .
 - ② $c_1(D, g)$ is a positive current.
- $\overline{D} = (D, g)$ is **integrable** $\iff \overline{D} = \overline{P} - \overline{Q}$ for some nef arithmetic \mathbb{R} -divisors \overline{P} and \overline{Q} .

Theorem (Arithmetic Hilbert-Samuel formula)

(Gillet-Soulé-Abbes-Bouche-Zhang-Moriwaki) If \overline{D} is nef, then

$$\hat{h}^0(X, n\overline{D}) = \frac{\widehat{\deg}(\overline{D}^d)}{d!} n^d + o(n^d).$$

In other words, $\widehat{\text{vol}}(\overline{D}) = \widehat{\deg}(\overline{D}^d)$.

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Remark

The above theorem suggests that $\widehat{\deg}(\overline{D}^d)$ can be defined by $\widehat{\text{vol}}(\overline{D})$. Note that

$$d!X_1 \cdots X_d = \sum_{I \subseteq \{1, \dots, d\}} (-1)^{d-\#(I)} \left(\sum_{i \in I} X_i \right)^d$$

in $\mathbb{Z}[X_1, \dots, X_d]$. Thus, for nef arithmetic \mathbb{R} -divisors $\overline{D}_1, \dots, \overline{D}_d$,

$$d! \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d) = \sum_{I \subseteq \{1, \dots, d\}} (-1)^{d-\#(I)} \widehat{\text{vol}} \left(\sum_{i \in I} \overline{D}_i \right).$$

In general, for integrable arithmetic \mathbb{R} -divisors $\overline{D}_1, \dots, \overline{D}_d$, we can define $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d)$ by linearity.

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Theorem (Generalized Hodge index theorem)

(Moriwaki) If \overline{D} is relatively nef, then $\widehat{\text{vol}}(\overline{D}) \geq \widehat{\text{deg}}(\overline{D}^d)$.

Corollary (The existence of small sections)

(Faltings-Gillet-Soulé-Zhang-Moriwaki) If \overline{D} is a relatively nef and $\widehat{\text{deg}}(\overline{D}^d) > 0$, then there are n and $\phi \in \text{Rat}(X)^\times$ such that $n\overline{D} + \widehat{(\phi)} \geq 0$.

Corollary (Arithmetic Bogomolov's inequality)

(Miyaoka-Soulé-Moriwaki) We assume $d = 2$ and X is regular. Let (E, h) be a C^∞ -hermitian locally free sheaf on X . If E is semistable on the generic fiber, then

$$\widehat{\text{deg}} \left(\widehat{c}_2(\overline{E}) - \frac{r-1}{2r} \widehat{c}_1(\overline{E})^2 \right) \geq 0,$$

where $r = \text{rk } E$.

Let $\pi : Y = \text{Proj} \left(\bigoplus_{n \geq 0} \text{Sym}^n(E) \right) \rightarrow X$ and D the tautological divisor on Y (i.e. $\mathcal{O}_Y(D) = \mathcal{O}(1)$). Roughly speaking, if we give a suitable Green function g to D , then $(D, g) - (1/r)\pi^*(\widehat{c}_1(\overline{E}))$ is relatively nef and its volume is zero, so that

$$\widehat{\text{deg}} \left(((D, g) - (1/r)\pi^*(\widehat{c}_1(\overline{E})))^{r+1} \right) \leq 0$$

by the Generalized Hodge index theorem, which gives the above inequality.

Theorem (Arithmetic Fujita's approximation theorem)

(Chen-Yuan) We assume that \overline{D} is big. For a given $\epsilon > 0$, there are a birational morphism $\nu_\epsilon : Y_\epsilon \rightarrow X$ of generically smooth, normal projective arithmetic varieties and a nef and big arithmetic \mathbb{Q} -divisor \overline{P} of C^∞ -type such that $\nu_\epsilon^*(\overline{D}) \geq \overline{P}$ and $\widehat{\text{vol}}(\overline{P}) \geq \widehat{\text{vol}}(\overline{D}) - \epsilon$.

Let S be a non-singular projective surface over an algebraically closed field. Let D be an effective divisor on S . By virtue of Bauer, the positive part of the Zariski decomposition of D is characterized by the greatest element of

$$\{M \mid M \text{ is a nef } \mathbb{R}\text{-divisor on } S \text{ and } M \leq D\}.$$

Let X be a d -dimensional, generically smooth normal projective arithmetic variety and let \overline{D} be a big arithmetic \mathbb{R} -divisor of C^0 -type on X . By the above theorem, a decomposition $\overline{D} = \overline{P} + \overline{N}$ is called a *Zariski decomposition of \overline{D}* if

- ① \overline{P} is a nef arithmetic \mathbb{R} -divisor on X .
- ② \overline{N} is an effective arithmetic \mathbb{R} -divisor of C^0 -type on X .
- ③ $\widehat{\text{vol}}(\overline{D}) = \widehat{\text{vol}}(\overline{P})$.

Let $\mathbb{P}_{\mathbb{Z}}^n = \text{Proj}(\mathbb{Z}[T_0, T_1, \dots, T_n])$, $D = \{T_0 = 0\}$ and $z_i = T_i/T_0$ for $i = 1, \dots, n$. Let us fix a positive number a . We define a D -Green function g_a of C^∞ -type on $\mathbb{P}^n(\mathbb{C})$ and an arithmetic divisor \overline{D}_a of C^∞ -type on $\mathbb{P}_{\mathbb{Z}}^n$ to be

$$g_a := \log(1 + |z_1|^2 + \dots + |z_n|^2) + \log(a) \quad \text{and} \quad \overline{D}_a := (D, g_a).$$

Note that $c_1(\overline{D}_a)$ is positive. Let

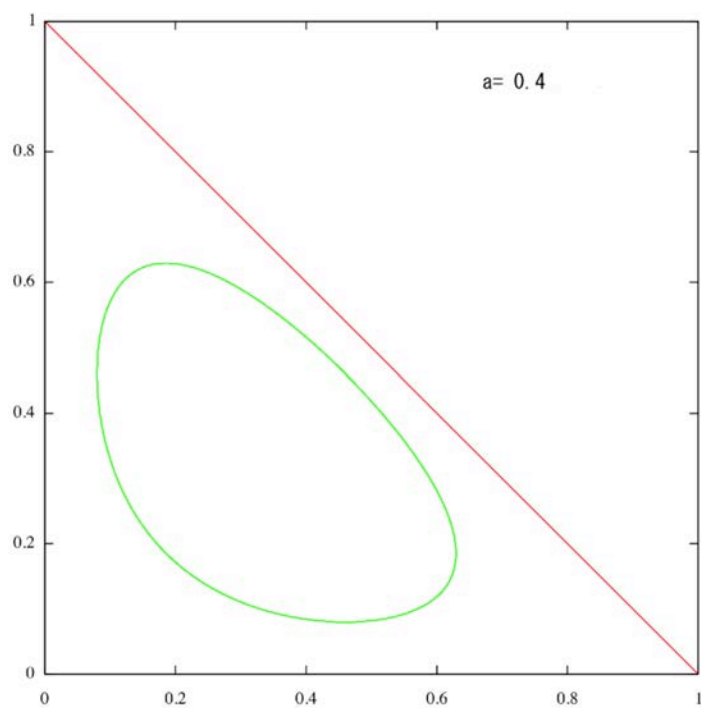
$$\Delta_n := \{(x_1, \dots, x_n) \in \mathbb{R}_{\geq 0}^n \mid x_1 + \dots + x_n \leq 1\}$$

and let $\vartheta_a : \Delta_n \rightarrow \mathbb{R}$ be a function given by

$$2\vartheta_g = -(1 - x_1 - \dots - x_n) \log(1 - x_1 - \dots - x_n) - \sum_{i=1}^n x_i \log x_i + \log(a).$$

We set

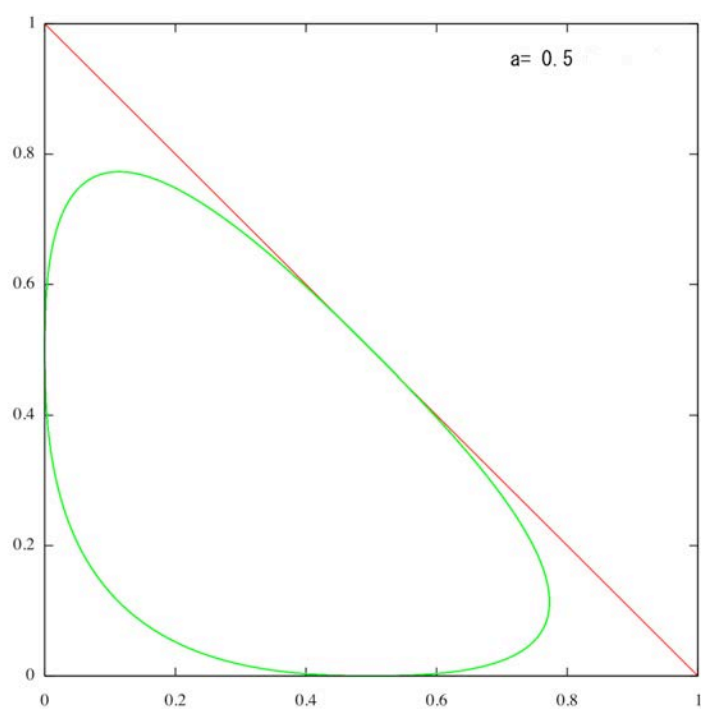
$$\Theta_a := \{(x_1, \dots, x_n) \in \Delta_n \mid \vartheta_a(x_1, \dots, x_n) \geq 0\}.$$



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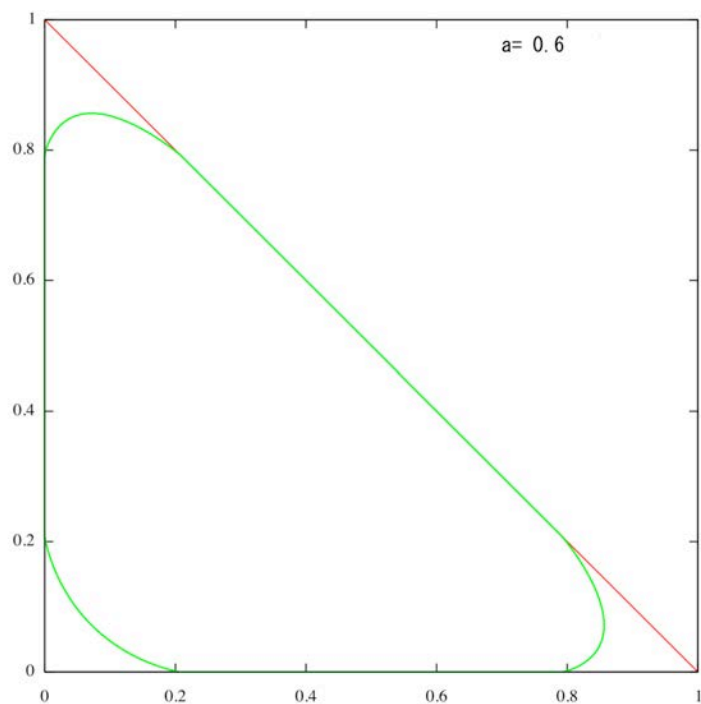
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The following properties (1) – (6) hold for \overline{D}_a :

- (1) \overline{D}_a is ample $\iff a > 1$.
- (2) \overline{D}_a is nef $\iff a \geq 1$.
- (3) \overline{D}_a is big $\iff a > \frac{1}{n+1}$.
- (4) \overline{D}_a is pseudo-effective $\iff a \geq \frac{1}{n+1}$.

(5) (Integral formula) The following formulae hold:

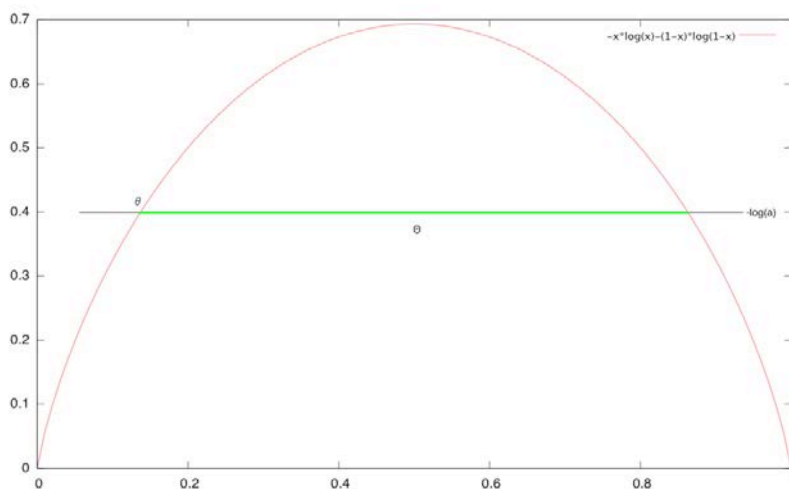
$$\widehat{\text{vol}}(\overline{D}_a) = (n+1)! \int_{\Theta_a} \vartheta_a(1-x_1-\cdots-x_n, x_1, \dots, x_n) dx_1 \cdots dx_n$$

and

$$\widehat{\text{deg}}(\overline{D}_a^{n+1}) = (n+1)! \int_{\Delta_n} \vartheta_a(1-x_1-\cdots-x_n, x_1, \dots, x_n) dx_1 \cdots dx_n.$$

Boucksom and H. Chen generalized the above formulae to a general situation by using Okounkov bodies.

(6) (Zariski decomposition for $n = 1$) We assume $n = 1$. The Zariski decomposition of \overline{D}_a exists if and only if $a \geq 1/2$. Moreover, we set $H_0 = D = \{T_0 = 0\}$, $H_1 = \{T_1 = 0\}$ and $\theta_a = \inf \Theta_a$.



If we set

$$p_a(z_1) = \begin{cases} \theta_a \log |z_1|^2 & \text{if } |z_1| < \sqrt{\frac{\theta_a}{1-\theta_a}}, \\ \log(1 + |z_1|^2) + \log(a) & \text{if } \sqrt{\frac{\theta_a}{1-\theta_a}} \leq |z_1| \leq \sqrt{\frac{1-\theta_a}{\theta_a}}, \\ (1 - \theta_a) \log |z_1|^2 & \text{if } |z_1| > \sqrt{\frac{1-\theta_a}{\theta_a}}, \end{cases}$$

then the positive part of \overline{D}_a is given by

$$((1 - \theta)H_0 - \theta H_1, p_a).$$

Let $\overline{D}_g = (H_0, g)$ be a big arithmetic \mathbb{R} -Cartier divisor of C^0 -type on $\mathbb{P}_{\mathbb{Z}}^n$. We assume that

$$g(\exp(2\pi\sqrt{-1}\theta_1)z_1, \dots, \exp(2\pi\sqrt{-1}\theta_n)z_n) = g(z_1, \dots, z_n)$$

for all $\theta_1, \dots, \theta_n \in [0, 1]$. We set

$$\xi_g(y_1, \dots, y_n) := \frac{1}{2}g(\exp(y_1), \dots, \exp(y_n))$$

for $(y_1, \dots, y_n) \in \mathbb{R}^n$. Let ϑ_g be the Legendre transform of ξ_g , that is,

$$\begin{aligned} \vartheta_g(x_1, \dots, x_n) \\ := \sup\{x_1 y_1 + \dots + x_n y_n - \xi_g(y_1, \dots, y_n) \mid (y_1, \dots, y_n) \in \mathbb{R}^n\} \end{aligned}$$

for $(x_1, \dots, x_n) \in \Delta_n$. Note that if $g = \log(1 + |z_1|^2 + \dots + |z_n|^2) + \log(a)$, then

$$2\vartheta_g = -(1 - x_1 - \dots - x_n) \log(1 - x_1 - \dots - x_n) - \sum_{i=1}^n x_i \log x_i + \log(a).$$

Theorem (Burgos Gil, Moriwaki, Philippon and Sombra)

There is a Zariski decomposition of $f^(\overline{D}_g)$ for some birational morphism $f : Y \rightarrow \mathbb{P}_{\mathbb{Z}}^n$ of generically smooth and projective normal arithmetic varieties if and only if*

$$\Theta_g := \{(x_1, \dots, x_n) \in \Delta_n \mid \vartheta_g(x_1, \dots, x_n) \geq 0\}$$

is a quasi-rational convex polyhedron, that is, there are $\gamma_1, \dots, \gamma_l \in \mathbb{Q}^n$ and $b_1, \dots, b_l \in \mathbb{R}$ such that

$$\Theta_g = \{x \in \mathbb{R}^n \mid \langle x, \gamma_i \rangle \geq b_i \ \forall i = 1, \dots, l\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product of \mathbb{R}^n .

The above theorem holds for toric varieties.

For example, if $g = \log \max\{a_0, a_1|z_1|^2, a_2|z_2|^2\}$, then \overline{D}_g is big if and only if $\max\{a_0, a_1, a_2\} > 1$. Moreover,

$$\Theta_g = \left\{ (x_1, x_2) \in \Delta_2 \mid \log\left(\frac{a_1}{a_0}\right)x_1 + \log\left(\frac{a_2}{a_0}\right)x_2 + \log(a_0) \geq 0 \right\}.$$

Thus there is a Zariski decomposition of $f^*(\overline{D}_g)$ for some birational morphism $f : Y \rightarrow \mathbb{P}_{\mathbb{Z}}^2$ of generically smooth and projective normal arithmetic varieties if and only if there is $\lambda \in \mathbb{R}_{>0}$ such that

$$\lambda \left(\log\left(\frac{a_1}{a_0}\right), \log\left(\frac{a_2}{a_0}\right) \right) \in \mathbb{Q}^2.$$